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The angular correlation between electron pairs created by γ-rays from the π^0-decay

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With 4 figures in the text

SUMMARY

The possibility of a direct determination of the intrinsic parity of the neutral π-meson by means of a bubble chamber experiment is investigated. The cross section for pair production by both the photons from the decay of a neutral π-meson is computed as a function of the angle ψ, defined in the following manner. Each positon-negaton pair defines a plane. ψ is the angle between the projections of these normals on a plane, perpendicular to the photon direction. The cross section has the form

$$\frac{d\sigma}{d\psi} = \alpha (1 \pm \beta^2 \cos 2\psi).$$

The sign is + for a scalar, − for a pseudoscalar meson. The computation gives $\beta^2 = 0.0091$. The Born approximation is used, and the effect of screening is neglected.

I. Introduction

It is generally assumed that both the charged and the neutral π-meson have odd intrinsic parity. This is experimentally verified for the charged π-meson, but not for the neutral one. The experiment used for the determination of the parity of the charged π-meson is the absorbtion of negative π-mesons in deuterium. From the fact that the reaction $\pi^- + D \to 2N$ occurs with great probability [1], one can show, using only conservation of parity and angular momentum, that the negative π-meson is pseudoscalar [2]. Of course, there is then every reason to believe that the positive π-meson also has odd intrinsic parity.

The argument used for the determination of the parity of the neutral π-meson is of an entirely different character. The frequency of occurrence of the two processes

$$\pi^- + P \to N + \pi^0; \quad \pi^0 \to 2\gamma \tag{1}$$

and

$$\pi^- + P \to N + \gamma \tag{2}$$

is measured with the result that the frequency is almost the same for both processes [1]. Using weak coupling theory arguments, one can conclude from this fact that the neutral π-meson has to be pseudoscalar, unless we assume a coupling constant which is very much larger than that which gives a reasonable strength for nuclear forces [3]. However, there is no reason why in meson theory the coupling constants defined from different experiments should be equal, or even of the same order of magnitude (cf. e.g. [4]), and the argument used to rule out even intrinsic parity for the neutral π-meson is thus very doubtful. A more direct experimental verification of the parity would therefore be very desirable. Such an experiment was first suggested by Yang [5]. The parity of a particle with spin zero, that decays into two quanta, can be determined from the relative polarization of the quanta. For even parity, the planes of polarization of the two quanta are parallel; for odd parity they are perpendicular to each other [6]. If both the decay photons produce positon-negaton pairs, the relative orientation of the polarization planes could in principle be observed by means of four counters, registering the four outgoing electrons. It turned out, however, that the experiment suggested by Yang was in its original form too difficult to be used practically for the determination of the parity. With the recent development of bubble chambers the experiment might be feasible, using a bubble chamber instead of the counters for the observation of the electrons. Therefore, we found it worth while to compute the cross sections for two-pair production from the two quanta emerging from a decaying particle with spin zero, and with odd or even intrinsic parity.

II. Derivation of the cross sections

The element of the S-matrix corresponding to two incoming photons with momenta k and k' and with polarizations e and e', and two outgoing positon-negaton pairs with momenta p_+, p_- and p'_+, p'_-, respectively, is written

$$\langle p_+, p_-; p'_+, p'_- | S | k, e; k', e' \rangle. \tag{3}$$

In the case that the photons are the two decay photons from a neutral π-meson, the state of polarization of the two-photon system depends on the parity of the π-meson in the following way. Let e_1 and e_2 be two orthogonal polarization directions. A state with one photon with momentum k and polarization e_1 is written $|k, e_1\rangle$. Then the two independent states of circularly polarized light are

$$|k, e^R\rangle = \frac{|k, e_1\rangle + i\,|k, e_2\rangle}{\sqrt{2}} \tag{4 a}$$

and

$$|k, e^L\rangle = \frac{|k, e_1\rangle - i\,|k, e_2\rangle}{\sqrt{2}}. \tag{4 b}$$

The total angular momentum of the system must be conserved in the decay process. As the decay of the π^0-meson into two quanta is the result of a strong interaction, we have every reason to believe that parity is conserved as well. Thus the two-photon state has to be a state with zero angular momentum, and with odd or even parity, depending on whether the intrinsic parity of the π^0-meson is odd or even. In the center-of-mass system of the meson, where the two photons move in opposite directions, the states with zero angular momentum are

$$|a\rangle = |k, e^R\rangle \, |k', e'^R\rangle \tag{5a}$$

and

$$|b\rangle = |k, e^L\rangle \, |k', e'^L\rangle. \tag{5b}$$

The following superpositions of these states have a definite parity [6]

$$|k, k', +\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle) = \frac{1}{\sqrt{2}}(|k, e^R\rangle \, |k', e'^R\rangle + |k, e^L\rangle \, |k', e'^L\rangle) \text{ (even)} \tag{6a}$$

and

$$|k, k', -\rangle = \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle) = \frac{1}{\sqrt{2}}(|k, e^R\rangle \, |k', e'^R\rangle - |k, e^L\rangle \, |k', e'^L\rangle) \text{ (odd).} \tag{6b}$$

Inserting (4) into (6) we get

$$|k, k', +\rangle = \frac{1}{\sqrt{2}}(|k, e_1\rangle \, |k', e_1'\rangle - |k, e_2\rangle \, |k', e_2'\rangle) \tag{7a}$$

and

$$|k, k', -\rangle = \frac{1}{\sqrt{2}}(|k, e_1\rangle \, |k', e_2'\rangle + |k, e_2\rangle \, |k', e_1'\rangle). \tag{7b}$$

The elements of the S-matrix for two-pair production thus become

$$\begin{aligned}&\langle p_+, p_-; p_+', p_-' \,|\, S \,|\, k, k', +\rangle \\ &= \frac{1}{\sqrt{2}}(\langle p_+, p_- \,|\, M \,|\, k, e_1\rangle \langle p_+', p_-' \,|\, M \,|\, k', e_1'\rangle - \langle p_+, p_- \,|\, M \,|\, k, e_2\rangle \langle p_+', p_-' \,|\, M \,|\, k', e_2'\rangle)\end{aligned} \tag{8a}$$

for the case of a scalar meson, and

$$\begin{aligned}&\langle p_+, p_-; p_+', p_-' \,|\, S \,|\, k, k', -\rangle \\ &= \frac{1}{\sqrt{2}}(\langle p_+, p_- \,|\, M \,|\, k, e_1\rangle \langle p_+', p_-' \,|\, M \,|\, k', e_2'\rangle + \langle p_+, p_- \,|\, M \,|\, k, e_2\rangle \langle p_+', p_-' \,|\, M \,|\, k', e_1'\rangle)\end{aligned} \tag{8b}$$

for the case of a pseudoscalar meson. Here $\langle p_+, p_- \,|\, M \,|\, k, e_1\rangle$ is the matrix element for pair production from a photon with momentum k and polarization e_1. The cross sections for two-pair production are then proportional to (for brevity we put $\langle p_+, p_- \,|\, M \,|\, k, e_1\rangle = M(e_1)$ etc.).

$$\begin{aligned}|\langle p_+, p_-; p_+', p_-' \,|\, S \,|\, k, k', +\rangle|^2 &= \tfrac{1}{2}\{|M(e_1)|^2 \, |M'(e_1')|^2 + |M(e_2)|^2 \, |M'(e_2')|^2 - \\ &- M^*(e_1) \, M'^*(e_1') \, M(e_2) \, M'(e_2') - M^*(e_2) \, M'^*(e_2') \, M(e_1) \, M'(e_1')\}\end{aligned} \tag{9a}$$

and

$$\begin{aligned}|\langle p_+, p_-; p_+', p_-' \,|\, S \,|\, k, k', -\rangle|^2 &= \tfrac{1}{2}\{|M(e_1)|^2 \, |M'(e_2')|^2 + |M(e_2)|^2 \, |M'(e_1')|^2 + \\ &+ M^*(e_2) \, M'^*(e_1') \, M(e_1) \, M'(e_2') + M^*(e_1) \, M'^*(e_2') \, M(e_2) \, M'(e_1')\}.\end{aligned} \tag{9b}$$

The matrix element for the production of a positon-negaton pair with momenta p_+ and p_- from a linearly polarized photon with momentum k and polarization $e_i\,(i=1,2)$ in an external field $A_\mu^{ext}(x)$ is given by

$$\langle p_+, p_- \,|\, M \,|\, k, e_i\rangle = \frac{i e^2}{V^{3/2}} \frac{1}{\sqrt{2\omega}} \bar{u}^{(r)}(p_-) \left[\gamma_\mu \frac{i\gamma(p_+-k)+m}{2p_+\cdot k} \gamma e_i + \gamma e_i \frac{i\gamma(p_--k)-m}{-2p_-\cdot k} \gamma_\mu\right] \times$$
$$\times u^{(r')}(-p_+) A_\mu^{ext}(q) \cdot 2\pi\delta(\omega-\varepsilon_+-\varepsilon_-) \qquad (10)$$

(cf. e.g. [7]), where $q=k-p_+-p_-$ and $A_\mu^{ext}(q)=\int dx\, A_\mu^{ext}(x)\, e^{iqx}$. It is evident from (10) that if we sum over the polarization directions of the outgoing electrons, $\langle p_+, p_- \,|\, M \,|\, k, e_i\rangle \langle p_+, p_- \,|\, M \,|\, k, e_j\rangle$ depends on the polarization vectors of the incoming photons only through the factor

$$Sp\left[\left(\gamma_\mu \frac{i\gamma(p_+-k)+m}{2p_+\cdot k}\gamma e_i + \gamma e_i \frac{i\gamma(p_--k)-m}{-2p_-\cdot k}\gamma_\mu\right)(i\gamma p_+ + m)\times\right.$$
$$\left.\times\left(\gamma e_j \frac{i\gamma(p_+-k)+m}{2p_+\cdot k}\gamma_\nu + \gamma_\nu \frac{i\gamma(p_--k)-m}{-2p_-\cdot k}\gamma e_j\right)(i\gamma p_- - m)\right]. \qquad (11)$$

The trace in (11) is a bilinear function of the polarization vectors e_i and e_j and thus the product of the matrix elements is of the form

$$M(e_i)\,M(e_j) = \sum_m (e_i\cdot A_m)\,(e_j\cdot B_m) + (e_i\cdot e_j)\,C, \qquad (12)$$

where A_m, B_m, and C are functions of the momenta. If we put $e_i=e_j=e_1$, we get

$$M(e_1)\,M(e_1) = \sum_m (e_1\cdot A_m)\,(e_1\cdot B_m) + C \qquad (13\,\text{a})$$

and if we put $e_i=e_1$, $e_j=e_2$

$$M(e_1)\,M(e_2) = \sum_m (e_1\cdot A_m)\,(e_2\cdot B_m).$$

In the case we are interested in, pair production by the Coulomb field from a nucleus with charge Ze, the cross section for pair production from a photon with polarization e is [8]

$$d\sigma = -\frac{\alpha Z^2}{(2\pi)^2} r_0^2 \frac{m^2}{(\bar{q}^2)^2} \frac{p_+ p_- d\varepsilon_+}{\omega} d\Omega_+ d\Omega_- \times$$
$$\times 2\left[\left(2\varepsilon_+ \frac{p_-\cdot e}{p_-\cdot k} + 2\varepsilon_- \frac{p_+\cdot e}{p_+\cdot k}\right)^2 - \bar{q}^2\left(\frac{p_-\cdot e}{p_-\cdot k} - \frac{p_+\cdot e}{p_+\cdot k}\right)^2 + \right.$$
$$\left. + \left(2 - \frac{\bar{q}^2\omega^2}{p_+\cdot k\, p_-\cdot k} + \frac{p_+\cdot k}{p_-\cdot k} + \frac{p_-\cdot k}{p_+\cdot k}\right)\right]. \qquad (14)$$

(14) tells us that $M(e_1)\,M(e_1)$ is of the form

$$M(e_1)\,M(e_1) = (\alpha\cdot e_1)^2 - (\beta\cdot e_1)^2 + \gamma \qquad (15\,\text{a})$$

and a comparison with (13) gives

$$M(e_1)\,M(e_2)=(\alpha\cdot e_1)(\alpha\cdot e_2)-(\beta\cdot e_1)(\beta\cdot e_2) \tag{15 b}$$

with the same α and β as in (15a). From (9), (14), and (15), we get the following expressions for the cross sections

$$\begin{aligned} d\sigma_s=&\frac{\alpha^2 Z^4}{(2\pi)^4} r_0^4\, m^4 \frac{p_+ p_- \, d\varepsilon_+}{\omega} d\Omega_+ \, d\Omega_- \times \frac{p'_+ p'_- \, d\varepsilon'_+}{\omega'} d\Omega'_+ \, d\Omega'_- \times \\ &\times 2\{[(A\cdot e_1)^2-(B\cdot e_1)^2+C]\times[(A'\cdot e'_1)^2-(B'\cdot e'_1)^2+C']+ \\ &+[(A\cdot e_2)^2-(B\cdot e_2)^2+C]\times[(A'\cdot e'_2)^2-(B'\cdot e'_2)^2+C']- \\ &-2[(\mathrm{A}\cdot e_1)(A\cdot e_2)-(B\cdot e_1)(B\cdot e_2)]\times[(A'\cdot e'_1)(A'\cdot e'_2)-(B'\cdot e'_1)(B'\cdot e'_2)]\} \end{aligned} \tag{16 a}$$

and

$$\begin{aligned} d\sigma_{ps}=&\frac{\alpha^2 Z^4}{(2\pi)^4} r_0^4\, m^4 \frac{p_+ p_- \, d\varepsilon_+}{\omega} d\Omega_+ \, d\Omega_- \times \frac{p'_+ p'_- \, d\varepsilon'_+}{\omega'} d\Omega'_- \, d\Omega'_- \times \\ &\times\{[(A\cdot e_2)^2-(B\cdot e_2)^2+C]\times[(A'\cdot e'_1)^2-(B'\cdot e'_1)^2+C']+ \\ &+[(A\cdot e_1)^2-(B\cdot e_1)^2+C]\times[(A'\cdot e'_2)^2-(B'\cdot e'_2)^2+C']+ \\ &+2[(A\cdot e_1)(A\cdot e_2)-(B\cdot e_1)(B\cdot e_2)]\times[(A'\cdot e'_1)(A'\cdot e'_2)-(B'\cdot e'_1)(B'\cdot e'_2)]\}, \end{aligned} \tag{16 b}$$

where

$$A=\frac{1}{\bar{q}^2}\left(2\,\varepsilon_+\frac{p_-}{p_-\cdot k}+2\,\varepsilon_-\frac{p_+}{p_+\cdot k}\right), \tag{16 c}$$

$$B=\frac{1}{\sqrt{\bar{q}^2}}\left(\frac{p_-}{p_-\cdot k}-\frac{p_+}{p_+\cdot k}\right), \tag{16 d}$$

$$C=\frac{1}{(\bar{q}^2)^2}\left(2-\frac{\bar{q}^2\,\omega^2}{p_+\cdot k\, p_-\cdot k}+\frac{p_+\cdot k}{p_-\cdot k}+\frac{p_-\cdot k}{p_+\cdot k}\right). \tag{16 e}$$

III. Introduction of coordinate systems and integration variables

We now introduce two coordinate systems, one for each positon-negaton pair. In both systems we use the direction of the incident photon as z-axis, and as x- and y-axes we use e_1, e_2, and e'_1, e'_2, respectively.

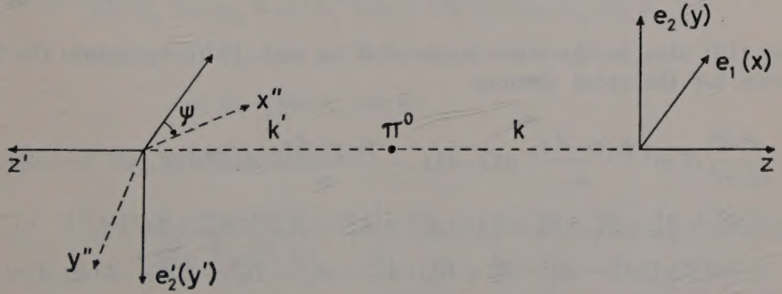

Fig. 1. The coordinate systems used in (17) and (21).

In these coordinate systems the expressions for the cross sections are

$$d\sigma_s = \frac{\alpha^2 Z^4}{(2\pi)^4} r_0^4 m^4 \frac{p_+ p_- d\varepsilon_+}{\omega} d\Omega_+ d\Omega_- \times \frac{p'_+ p'_- d\varepsilon'_+}{\omega'} d\Omega'_+ d\Omega'_- \times$$
$$\times 2\{(A_x A'_{x'} - A_y A'_{y'})^2 - (A_x B'_{x'} - A_y B'_{y'})^2 - (A'_{x'} B_x - A'_{y'} B_y)^2 +$$
$$+ (B_x B'_{x'} - B_y B'_{y'})^2 + C[A'^2_{x'} + A'^2_{y'} - B'^2_{x'} - B'^2_{y'}] +$$
$$+ C'[A_x^2 + A_y^2 - B_x^2 - B_y^2] + 2CC'\} \tag{17 a}$$

and

$$d\sigma_{ps} = \frac{\alpha^2 Z^4}{(2\pi)^4} r_0^4 m^4 \frac{p_+ p_- d\varepsilon_+}{\omega} d\Omega_+ d\Omega_- \times \frac{p'_+ p'_- d\varepsilon'_+}{\omega'} d\Omega'_+ d\Omega'_- \times$$
$$\times 2\{(A_x A'_{y'} + A_y A'_{x'})^2 - (A_x B'_{y'} + A_y B'_{x'})^2 - (B_x A'_{y'} + B_y A'_{x'})^2 +$$
$$+ (B_x B'_{y'} + B_y B'_{x'})^2 + C[A'^2_{x'} + A'^2_{y'} - B'^2_{x'} - B'^2_{y'}] +$$
$$+ C'[A_x^2 + A_y^2 - B_x^2 - B_y^2] + 2CC'\}. \tag{17 b}$$

We now introduce another coordinate system for the primed quantities, which is rotated an angle ψ around the z'-axis. The transformation is given by

$$\left.\begin{aligned} A'_{x'} &= A'_{x''} \cos\psi + A'_{y''} \sin\psi, \\ A'_{y'} &= -A'_{x''} \sin\psi + A'_{y''} \cos\psi. \end{aligned}\right\} \tag{18}$$

From (18) we get

$$(A_x A'_{x'} - A_y A'_{y'})^2 = \tfrac{1}{2}(A_x^2 + A_y^2)(A'^2_{x''} + A'^2_{y''}) +$$
$$+ \tfrac{1}{2}\cos 2\psi[(A_x^2 - A_y^2)(A'^2_{x''} - A'^2_{y''}) - 4A_x A_y A'_{x''} A'_{y''}] +$$
$$+ \sin 2\psi[(A_x^2 - A_y^2) A'_{x''} A'_{y''} + (A'^2_{x''} - A'^2_{y''}) A_x A_y] \tag{19}$$

and

$$(A_x A'_{y'} + A_y A'_{x'})^2 = \tfrac{1}{2}(A_x^2 + A_y^2)(A'^2_{x''} + A'^2_{y''}) -$$
$$- \tfrac{1}{2}\cos 2\psi[(A_x^2 - A_y^2)(A'^2_{x''} - A'^2_{y''}) - 4A_x A_y A'_{x''} A'_{y''}] -$$
$$- \sin 2\psi[(A_x^2 - A_y^2) A'_{x''} A'_{y''} + (A'^2_{x''} - A'^2_{y''}) A_x A_y]. \tag{20}$$

Inserting (18) also in the other terms of (17 a) and (17 b), we obtain the following expressions for the cross sections

$$d\sigma_{\substack{s\\ps}} = \frac{\alpha^2 Z^4}{(2\pi)^4} r_0^4 m^4 \frac{p_+ p_- d\varepsilon_+}{\omega} d\Omega_+ d\Omega_- \times \frac{p'_+ p'_- d\varepsilon'_+}{\omega'} d\Omega'_- d\Omega'_- \times$$
$$\times \{(A_x^2 + A_y^2 - B_x^2 - B_y^2 + 2C)(A'^2_{x''} + A'^2_{y''} - B'^2_{x''} - B'^2_{y''} + 2C') \pm$$
$$\pm \cos 2\psi[(A_x^2 - A_y^2 - B_x^2 + B_y^2)(A'^2_{x''} - A'^2_{y''} - B'^2_{x''} + B'^2_{y''}) - 4(A_x A_y - B_x B_y) \times$$
$$\times (A'_{x''} A'_{y''} - B'_{x''} B'_{y''})] \pm 2 \sin 2\psi[(A_x^2 - A_y^2 - B_x^2 + B_y^2)(A'_{x''} A'_{y''} - B'_{x''} B'_{x''}) +$$
$$+ (A'^2_{x''} - A'^2_{y''} - B'^2_{x''} + B'^2_{y''})(A_x A_y - B_x B_y)]\}. \tag{21}$$

The following system of angles is used to define the directions of the outgoing electrons p_+, p_-. Let ϑ be the angle between the direction of the incident photon and the direction of the normal $\hat{n}$ to the plane, defined by the outgoing pair. The angle between the projection of $\hat{n}$ on the $x-y$-plane and the x-axis is called φ. u_+ and u_- are the angles between the line where the plane defined by the outgoing pair cuts the $x-y$-plane and the direction of the outgoing positon and negaton, respectively. The corresponding dashed quantities are defined in the same way for the $p'_+ - p'_-$-system.

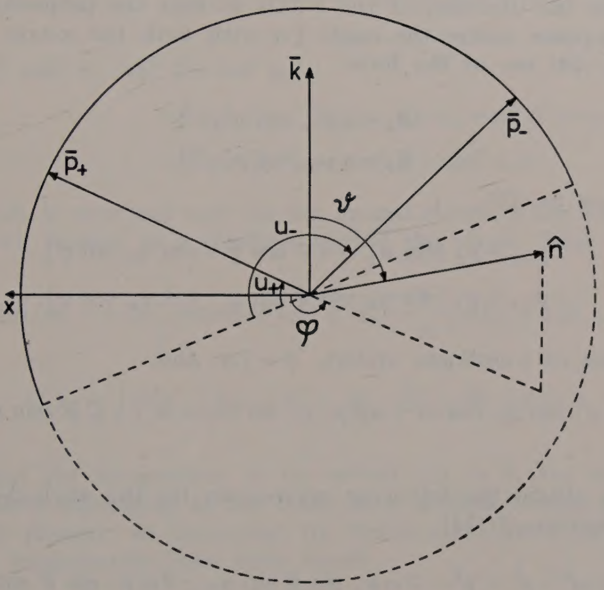

Fig. 2. The system of angles used in the integration of the cross section.

The formulae effecting the transformation from the system of polar angles $(\theta_+, \varphi_+, \theta_-, \varphi_-)$ to this system of angles are

$$\begin{aligned} \sin\theta_\pm \cos\varphi_\pm &= \sin u_\pm \cos\vartheta \cos\varphi + \cos u_\pm \sin\varphi, \\ \sin\theta_\pm \sin\varphi_\pm &= \sin u_\pm \cos\vartheta \sin\varphi - \cos u_\pm \cos\varphi, \\ \cos\theta_\pm &= \sin u_\pm \sin\vartheta. \end{aligned} \tag{22}$$

The Jacobian of the transformation is

$$\left| \frac{\partial\,(\cos\theta_+, \varphi_+; \cos\theta_-, \varphi_-)}{\partial\,(u_+, u_-, \vartheta, \varphi)} \right| = |\sin(u_+ - u_-)| \cdot |\sin\vartheta|. \tag{23}$$

We shall carry out an integration over all the angles, except the relative azimuth of the normals. As the different systems of pairs are separated in (21), we see that the integrations to be carried out are

$$\left.\begin{aligned}
&I_i=\int_0^{2\pi} d u_+ \int_0^{2\pi} d u_- \left|\sin (u_+ - u_-)\right| \int_0^{\pi} \sin \vartheta \, d \vartheta \times G_i \qquad (i=1,\,2,\,3),\\
&G_1=A_x^2+A_y^2-B_x^2-B_y^2+2\,C,\\
&G_2=A_x^2-A_y^2-B_x^2+B_y^2,\\
&G_3=2\,(A_x A_y-B_x B_y).
\end{aligned}\right\} \qquad (24)$$

We now choose the direction of the x-axis so that the projection of the normal $\hat{n}$ on the $x-y$-plane makes the angle $\frac{1}{4}\pi$ with both the x-axis and the y-axis. A_x and A_y in (24) are of the form

$$\left.\begin{aligned}
A_x&=\alpha\, p_{+x}+\beta\, p_{-x}\,,\\
A_y&=\alpha\, p_{+y}+\beta\, p_{-y}\,,
\end{aligned}\right\} \qquad (25)$$

where, according to (22),

$$\begin{aligned}
p_{\pm x}&=p_{\pm}\,[\sin u_{\pm}\,\cos\vartheta\,\cos\varphi+\cos u_{\pm}\,\sin\varphi],\\
p_{\pm y}&=p_{\pm}\,[\sin u_{\pm}\,\cos\vartheta\,\sin\varphi-\cos u_{\pm}\,\cos\varphi].
\end{aligned} \qquad (26)$$

With our choice of coordinate system, $\varphi=\frac{1}{4}\pi$, and

$$A_x^2-A_y^2=2\,(\alpha^2 p_+^2\,\sin u_+\,\cos u_+ +\alpha\beta\, p_+ p_-\,\sin(u_+ + u_-)+\beta^2 p_-^2\,\sin u_-\,\cos u_-)\,\cos\vartheta. \qquad (27)$$

From (22) we obtain the following expressions for the angle-dependent factors in the denominators of (14)

$$\left.\begin{aligned}
\bar{q}^2&=\omega^2+p_+^2+p_-^2-2\,\omega\, p_+\,\sin\vartheta\,\sin u_+ -2\,\omega\, p_-\,\sin\vartheta\,\sin u_- +\\
&\quad +2\, p_+ p_-\,\cos(u_+ - u_-),\\
p_+\cdot k&=-\varepsilon_+\,\omega\,(1-\beta_+\,\sin\vartheta\,\sin u_+),\\
p_-\cdot k&=-\varepsilon_-\,\omega\,(1-\beta_-\,\sin\vartheta\,\sin u_-).
\end{aligned}\right\} \qquad (28)$$

It is clear from (28) that α and β depend on ϑ only in being rational functions of $\sin\vartheta$, and, as is seen from (27), the integration over ϑ in I_2 takes the form

$$\int_0^{\pi} \sin\vartheta\, d\vartheta\, \cos\vartheta\, f(\sin\vartheta)=0. \qquad (29)$$

Thus I_2 is equal to zero, and we have only to compute I_1 and I_3. Alternatively, we can arrive at the conclusion $I_2=0$ in a more physical way. The cross sections for pair production from a photon with polarization along the x-axis, and a photon with polarization along the y-axis are proportional to, respectively,

$$\left.\begin{aligned}
\sigma_x&\sim A_x^2-B_x^2+C,\\
\sigma_y&\sim A_y^2-B_y^2+C,
\end{aligned}\right\} \qquad (30)$$

and it is immediately clear from symmetry reasons that the integrated cross sections are equal, what gives $I_2=0$. In order to get the same result for the primed quantities, ψ has to be chosen to be the relative azimuth of the normals.

With our choice of coordinate system we also get

$$2\,A_x\,A_y=\alpha^2\,p_+^2\,(\sin^2 u_+\,\cos^2\vartheta-\cos^2 u_+)+ \\ +2\,\alpha\,\beta\,p_+\,p_-\,(\sin u_+\,\sin u_-\,\cos^2\vartheta-\cos u_+\,\cos u_-) \\ +\beta^2\,p_-^2\,(\sin^2 u_-\,\cos^2\vartheta-\cos^2 u_-) \tag{31}$$

and

$$A_x^2+A_y^2=\alpha^2\,p_+^2\,(\sin^2 u_+\,\cos^2\vartheta+\cos^2 u_+)+ \\ +2\,\alpha\,\beta\,p_+\,p_-\,(\sin u_+\,\sin u_-\,\cos^2\vartheta+\cos u_+\,\cos u_-)+ \\ +\beta^2\,p_-^2\,(\sin^2 u_-\,\cos^2\vartheta+\cos^2 u_-). \tag{32}$$

The last formula is true not only for this special choice of the x-axis, as $A_x^2+A_y^2$ is invariant with respect to a rotation around the z-axis. Thus

$$\int_0^{2\pi} d\,u_+\int_0^{2\pi} d\,u_-\,|\sin(u_+-u_-)|\int_0^{\pi}\sin\vartheta\,d\,\vartheta\,(A_x^2+A_y^2)=\frac{1}{\pi}\int d\,\Omega_+\int d\,\Omega_-\,(A_x^2+A_y^2) \tag{33}$$

$$I_1=\frac{1}{\pi}\int d\,\Omega_+\int d\,\Omega_-\,(A_x^2+A_y^2-B_x^2-B_y^2+2\,C). \tag{34}$$

That means that the integrations to be carried out in I_1 are exactly the same as the integrations appearing in the total cross section for pair production from an unpolarized photon, as computed by Bethe and Heitler [9]. Thus we can write down I_1 immediately from their result

$$I_1=-\frac{4\,\pi}{m^2\,\omega^2}\left\{-\frac{4}{3}-\frac{2\,\varepsilon_+\,\varepsilon_-}{p_-^2}-\frac{2\,\varepsilon_+\,\varepsilon_-}{p_+^2}+\frac{\varepsilon_+\,m^2}{p_-^3}\,l_-+\frac{\varepsilon_-\,m^2}{p_+^3}\,l_+-\frac{m^2}{p_+\,p_-}\,l_+\cdot l_-+\right. \\ +L\left[\frac{\omega^2\,\varepsilon_+^2\,\varepsilon_-^2}{p_+^3\,p_-^3}+\frac{\omega^2}{p_+\,p_-}-\frac{8\,\varepsilon_+\,\varepsilon_-}{3\,p_+\,p_-}-\frac{\varepsilon_+\,\varepsilon_-\,\omega\,m^2}{2\,p_+\,p_-^4}\,l_-+\right. \\ \left.\left.+\frac{\omega\,m^2}{2\,p_+\,p_-^2}\,l_--\frac{\varepsilon_+\,\varepsilon_-\,\omega\,m^2}{2\,p_+^4\,p_-}\,l_++\frac{\omega\,m^2}{2\,p_+^2\,p_-}\,l_+-\frac{m^2\,\omega^2\,\varepsilon_+\,\varepsilon_-}{p_+^3\,p_-^3}\right]\right\}; \tag{35}$$

$$l_\pm=2\log\frac{\varepsilon_\pm+p_\pm}{m};\quad L=2\,\log\frac{\varepsilon_+\,\varepsilon_-+p_+\,p_-+m^2}{\omega\,m}.$$

From (31) and (32) we get

$$2\,A_x\,A_y=A_x^2+A_y^2-2\,(\alpha^2\,p_+^2\,\cos^2 u_++2\,\alpha\,\beta\,p_+\,p_-\,\cos u_+\,\cos u_-+\beta^2\,p_-^2\,\cos^2 u_-) \tag{36}$$

and thus

$$I_3=I_3'-I_3'', \tag{37 a}$$

where

$$I_3'=\frac{1}{\pi}\int d\,\Omega_+\int d\,\Omega_-\,(A_x^2+A_y^2-B_x^2-B_y^2) \tag{37 b}$$

and

$$I_3'' = 2\int_0^{2\pi} d u_+ \int_0^{2\pi} d u_- \,|\sin(u_+ - u_-)|\int_0^{\pi} \sin\vartheta\, d\vartheta \times$$

$$\times [(\alpha^2-\gamma^2)\,p_+^2\cos^2 u_+ + 2(\alpha\beta-\gamma\delta)\,p_+\,p_-\cos u_+\cos u_- + (\beta^2-\delta^2)\,p_-^2\cos^2 u_-]. \quad (37\,\text{c})$$

Here γ and δ are given by $B_x = \gamma\, p_{+x} + \delta\, p_{-x}$; $B_y = \gamma\, p_{+y} + \delta\, p_{-y}$. If we summarize the results of the analysis above, we see that the cross section for two-pair production expressed as a function of the angle ψ between the projections, on a plane perpendicular to the photon direction, of the normals to the planes defined by the outgoing pair is of the form

$$d\sigma = \alpha(1 \pm \beta^2 \cos 2\psi)\, d\psi, \quad (38)$$

where the sign is plus for a scalar meson, minus for a pseudoscalar. The value of β is

$$\beta = \frac{I_3' - I_3''}{I_1}, \quad (39)$$

where I_1 is given by (35),

$$I_3' = \frac{1}{\pi}\int d\Omega_+ \int d\Omega_- \times$$

$$\times \left[\frac{1}{(\bar{q}^2)^2}\left(4\,\varepsilon_+^2 \frac{p_{-x}^2 + p_{-y}^2}{(p_-\cdot k)^2} + 4\,\varepsilon_-^2\,\frac{p_{+x}^2 + p_{+y}^2}{(p_+\cdot k)^2} + 8\,\varepsilon_+\,\varepsilon_-\,\frac{p_{+x}\,p_{-x} + p_{+y}\,p_{-y}}{p_+\cdot k\, p_-\cdot k}\right) - \right.$$

$$\left. - \frac{1}{\bar{q}^2}\left(\frac{p_{-x}^2 + p_{-y}^2}{(p_-\cdot k)^2} + \frac{p_{+x}^2 + p_{+y}^2}{(p_+\cdot k)^2} - 2\,\frac{p_{+x}\,p_{-x} + p_{+y}\,p_{-y}}{p_+\cdot k\, p_-\cdot k}\right)\right] \quad (40)$$

and

$$I_3'' = 2\int_0^{2\pi} d u_+ \int_0^{2\pi} d u_- \,|\sin(u_+ - u_-)|\int_0^{\pi} \sin\vartheta\, d\vartheta \times$$

$$\times \left[\frac{1}{(\bar{q}^2)^2}\left(4\,\varepsilon_+^2 \frac{p_-^2\cos^2 u_-}{(p_-\cdot k)^2} + 4\,\varepsilon_-^2\,\frac{p_+^2\cos^2 u_+}{(p_+\cdot k)^2} + 8\,\varepsilon_+\,\varepsilon_-\,\frac{p_+\,p_-\cos u_+\cos u_-}{p_+\cdot k\, p_-\cdot k}\right) - \right.$$

$$\left. - \frac{1}{\bar{q}^2}\left(\frac{p_-^2\cos^2 u_-}{(p_-\cdot k)^2} + \frac{p_+^2\cos^2 u_+}{(p_+\cdot k)^2} - 2\,\frac{p_+\,p_-\cos u_+\cos u_-}{p_+\cdot k\, p_-\cdot k}\right)\right]. \quad (41)$$

IV. Computation of I_3'

The term I_3' is part of the Bethe-Heitler cross section, and is thus relatively easy to compute. The most convenient way of carrying out the integrations is the method used by Gluckstern and Hull [10]. We illustrate their method by computing the first three terms of I_3'. In the first term the angles associated with the positon enter only in q^2, and we can carry out an integration over these angles first. To do this, we write

$$\bar{q}^2=(T^2+p_+^2)\,(1-\bar{p}_+\cdot\bar{a}), \tag{42}$$

where $\bar{a}=2\,\bar{T}/(T^2+p_+^2)$ and $\bar{T}=\bar{k}-\bar{p}_-$. Thus

$$\int d\Omega_+\frac{1}{(\bar{q}^2)^2}=\frac{1}{(T^2+p_+^2)^2}\int d\Omega_+\frac{1}{(1-\bar{p}_+\cdot\bar{a})^2}=\frac{2\pi}{(T^2+p_+^2)^2}\int\limits_{-1}^{+1} d\,(\cos\theta_{p_+\,a})\cdot$$

$$\frac{1}{(1-p_+\,a\cos\theta_{p_+\,a})^2}=\frac{4\pi}{T^2-p_+^2}=\frac{\pi}{\omega^2\,\varepsilon_-^2\,(1-\beta_-\,\cos\,\theta_-)^2}. \tag{43}$$

The first term of I_3' is thus equal to

$$\frac{4\,\varepsilon_+^2\,p_-^2\cdot 2\,\pi}{\pi\cdot\omega^4\,\varepsilon_-^4}\int\limits_{-1}^{+1} d\,(\cos\,\theta_-)\,\frac{\sin^2\theta_-}{(1-\beta_-\,\cos\,\theta_-)^4}=\frac{32\,\varepsilon_+^2\,p_-^2}{3\,m^4\,\omega^4}. \tag{44}$$

To get the second term of I_3' we have only to change plus into minus and vice versa in (44)

$$\frac{4\,\varepsilon_-^2}{\pi}\int d\Omega_+\int d\Omega_-\frac{1}{(\bar{q}^2)^2}\,\frac{p_{+x}^2+p_{+y}^2}{(p_+\cdot k)^2}=\frac{32\,\varepsilon_-^2\,p_+^2}{3\,m^4\,\omega^4}. \tag{45}$$

In the third term of I_3' we use (42) and

$$p_+\cdot k=-\,\varepsilon_+\,\omega\,(1-\bar{p}_+\cdot\bar{b}), \tag{46}$$

where $\bar{b}=\bar{k}/E_+\,\omega$. The expression $p_{+x}\,p_{-x}+p_{+y}\,p_{-y}$ may be thought of as the scalar product of $\bar{p}_+$ with $\bar{p}_-'$, the projection of $\bar{p}_-$ on the plane perpendicular to $\bar{k}$. $\bar{p}_-'$ lies in the $\bar{p}_- - \bar{k}$-plane, and can thus be expressed as a linear combination of $\bar{p}_-$ and $\bar{k}$, or of $\bar{a}$ and $\bar{b}$.

$$\bar{p}_-'=m\,\bar{a}+n\,\bar{b}. \tag{47}$$

The conditions $\bar{p}_-\cdot\bar{k}=0$ and $\bar{p}_-'^2=p_-^2\,\sin^2\,\theta_-$ determine the absolute values of m and n. The signs can be determined by the condition $\bar{p}_-\cdot\bar{p}_-'>0$. The expressions arrived at are

$$m=-\frac{T^2+p_+^2}{2} \tag{48 a}$$

and

$$n=\varepsilon_+\,(\omega-p_-\,\cos\,\theta_-). \tag{48 b}$$

Thus
$$p_{+x}\,p_{-x}+p_{+y}\,p_{-y}=-\,m\,(1-\bar{p}_+\cdot\bar{a})-n\,(1-\bar{p}_+\cdot\bar{b})+m+n \tag{49}$$

and the term of I_3' under consideration can by the integration over $d\,\Omega_+$, which we carry out first, be split up into three parts

$$\int d\Omega_+\,\frac{1}{(\bar{q}^2)^2}\,\frac{p_{+x}\,p_{-x}+p_{+y}\,p_{-y}}{p_+\cdot k}=\frac{m}{(T^2\,+\,p_+^2)^2\,\varepsilon_+\,\omega}\int d\,\Omega_+\,\frac{1}{(1-\bar{p}_+\cdot\bar{a})\,(1-\bar{p}_+\cdot\bar{b})}+$$

$$+\frac{n}{(T^2+p_+^2)^2\,\varepsilon_+\,\omega}\int d\Omega_+\frac{1}{(1-\bar{p}_+\cdot\bar{a})^2}-$$

$$-\frac{m+n}{(T^2+p_+^2)^2\,\varepsilon_+\,\omega}\int d\Omega_+\frac{1}{(1-\bar{p}_+\cdot\bar{a})^2\,(1-\bar{p}_+\cdot\bar{b})} \qquad (50)$$

The terms of (50) can now be integrated in the following way. The integral in the first term is rewritten with the aid of the identity [11]

$$(\alpha\beta)^{-1}=\int_0^1 dx\,[\alpha x+\beta(1-x)]^{-2} \qquad (51)$$

what gives

$$\int d\Omega_+\frac{1}{(1-\bar{p}_+\cdot\bar{a})\,(1-\bar{p}_+\cdot\bar{b})}=\int_0^1 dx\int d\Omega_+\frac{1}{(1-\bar{p}_+\cdot\bar{g})^2}, \qquad (52)$$

where $\bar{g}=x\bar{a}+(1-x)\bar{b}$. If $\bar{g}$ is now used as the polar axis, we get

$$\int d\Omega_+\frac{1}{(1-\bar{p}_+\cdot\bar{a})\,(1-\bar{p}_+\cdot\bar{b})}=2\pi\int_0^1 dx\int_{-1}^{+1} d\,(\cos\theta_{p_+\,g})\,\frac{1}{(1-p_+\,g\cos\theta_{p_+\,g})^2}$$

$$=4\pi\int_0^1\frac{dx}{X}, \qquad (53)$$

where $X=1-p_+^2\,g^2$ is a quadratic in x. If we carry put the x-integration we get

$$\int d\Omega_+\frac{1}{(1-\bar{p}_+\cdot\bar{a})\,(1-\bar{p}_+\cdot\bar{b})}=\frac{\pi\,(T^2+p_+^2)\,\varepsilon_+}{\varepsilon_-\,p_+\,p_-\,(1-\beta_-\cos\theta_-)}\cdot L, \qquad (54)$$

where L is the quantity introduced in (35). The integration over the remaining variables in the term considered is now elementary and straightforward. The other terms of I_3' are treated in an analogous way, and we give only the result here

$$I_3'=\frac{4\pi}{m^2\,\omega^2}\left\{-\frac{8}{3}-2\frac{m^2}{p_+^2}-2\frac{m^2}{p_-^2}-\frac{\varepsilon_+\,m^2}{p_+^3}\,l_+-\frac{\varepsilon_-\,m^2}{p_-^3}\,l_-+\frac{m^2}{p_+\,p_-}\,l_+\cdot l_-+\right.$$

$$+L\left[\frac{8\,\varepsilon_+\,\varepsilon_-}{3\,p_+\,p_-}+\frac{\varepsilon_+\,\omega\,m^2}{p_+^3\,p_-}+\frac{\varepsilon_-\,\omega\,m^2}{p_+\,p_-^3}-\right.$$

$$\left.\left.-\frac{m^2\,\omega}{p_+^2\,p_-}\,l_+-\frac{m^4\,\omega}{2\,p_+^4\,p_-}\,l_+-\frac{m^2\,\omega}{p_+\,p_-^2}\,l_--\frac{m^4\,\omega}{2\,p_+\,p_-^4}\,l_-\right]\right\}. \qquad (55)$$

V. Compution of I_3''

What remains is to compute the second part of I_3, that is to compute the following six integrals:

$$G_i=\int_0^{2\pi} d u_+ \int_0^{2\pi} d u_- \,|\sin(u_+-u_-)|\int_0^{\pi}\sin\vartheta\, d\vartheta\times F_i \quad (i=1,\,2,\,3,\,4,\,5,\,6), \tag{56 a}$$

$$F_1=\frac{\cos^2 u_-}{(\bar{q}^2)^2(p_-\cdot k)^2}, \tag{56 b}$$

$$F_2=\frac{\cos^2 u_+}{(\bar{q}^2)^2(p_+\cdot k)^2}, \tag{56 c}$$

$$F_3=\frac{\cos u_+ \cos u_-}{(\bar{q}^2)^2\, p_+\cdot k\, p_-\cdot k}, \tag{56 d}$$

$$F_4=\frac{\cos^2 u_-}{\bar{q}^2(p_-\cdot k)^2}, \tag{56 e}$$

$$F_5=\frac{\cos^2 u_+}{\bar{q}^2(p_+\cdot k)^2}, \tag{56 f}$$

$$F_6=\frac{\cos u_+ \cos u_-}{\bar{q}^2\, p_+\cdot k\, p_-\cdot k}. \tag{56 g}$$

Of course, we get F_2 and F_5 from F_1 and F_4 respectively, by everywhere interchanging the role of plus and minus. Thus we have only four essentially different terms to compute. To compute G_1, we introduce the angle $u=u_- - u_+$, and use u_-, u, and ϑ as integration variables. We write

$$\bar{q}^2=\alpha_1(1+\beta_1'\cos u_- +\gamma_1'\sin u_-), \tag{57 a}$$

$$-p_-\cdot k=\alpha_2(1+\beta_2'\cos u_- +\gamma_2'\sin u_-), \tag{57 b}$$

where

$$\beta_i',\ \gamma_i'=\beta_i\sin\vartheta,\ \gamma_i\sin\vartheta; \tag{57 c}$$

$$\alpha_1=\omega^2+p_+^2+p_-^2+2\,p_+\,p_-\cos u;\quad \alpha_1\beta_1=2\,\omega\, p_+\sin u;\quad \alpha_1\gamma_1=-2\,\omega\,(p_-+p_+\cos u); \tag{57 d}$$

$$\alpha_2=\varepsilon_-\,\omega;\quad \beta_2=0;\quad \gamma_2=-\beta_-. \tag{57 e}$$

The integral over u_- now takes the form

$$\int_0^{2\pi} d u_- \frac{\cos^2 u_-}{(1+\beta_1'\cos u_- +\gamma_1'\sin u_-)^2(1+\beta_2'\cos u_- +\gamma_2'\sin u_-)^2}$$

$$=\frac{\partial^2}{\partial\beta_1'\partial\beta_2'}\int_0^{2\pi} d u_- \frac{1}{(1+\beta_1'\cos u_- +\gamma_1'\sin u_-)(1+\beta_2'\cos u_- +\gamma_2'\sin u_-)}. \tag{58}$$

Using the identity (51) we have

$$\int_0^{2\pi} \frac{d u_-}{(1+\beta_1' \cos u_- + \gamma_1' \sin u_-)(1+\beta_2' \cos u_- + \gamma_2' \sin u_-)}$$

$$= \int_0^1 d x \int_0^{2\pi} \frac{d u_-}{(1+\beta' \cos u_- + \gamma' \sin u_-)^2} = 2\pi \int_0^1 d x \frac{1}{(1-\beta'^2-\gamma'^2)^{3/2}} \tag{59}$$

with $\beta' = \beta_1' x + \beta_2' (1-x)$; $\gamma' = \gamma_1' x + \gamma_2' (1-x)$. The most convenient way to continue the computation is now first to carry out one of the differentiations in (58), say the one with respect to β_1', and then carry out the integration over ϑ

$$\int_0^{\pi} \sin\vartheta\, d\vartheta \int_0^{2\pi} d u_- \frac{\cos^2 u_-}{(1+\beta_1' \cos u_+ + \gamma_1' \sin u_-)^2 (1+\beta_2' \cos u_- + \gamma_2' \sin u_-)^2}$$

$$= 6\pi \int_0^{\pi} \sin\vartheta\, d\vartheta \frac{\partial}{\partial \beta_2} \int_0^1 d x \frac{\beta}{[1-(\beta^2+\gamma^2)\sin^2\vartheta]^{5/2}}$$

$$= 4\pi \frac{\partial}{\partial \beta_2} \int_0^1 d x \left[\frac{\beta x}{1-\beta^2-\gamma^2} + \frac{2\beta x}{(1-\beta^2-\gamma^2)^2}\right], \tag{60}$$

where $\beta' = \beta \sin\vartheta$, $\gamma' = \gamma \sin\vartheta$. From (57) and (60) we now get

$$\int_0^{\pi} \sin\vartheta\, d\vartheta \int_0^{2\pi} d u_- \cdot F_1 = \frac{4\pi}{\alpha_1^2 \alpha_2^2} \int_0^1 d x (1-x) x \times$$

$$\left[\frac{1}{1-\beta^2-\gamma^2} + \frac{2}{(1-\beta^2-\gamma^2)^2} + \frac{2\beta^2}{(1-\beta^2-\gamma^2)^2} + \frac{8\beta^2}{(1-\beta^2-\gamma^2)^3}\right]. \tag{61}$$

The expression $1-\beta^2-\gamma^2$ in the denominators of (61) is quadratic in x. Its explicit form is

$$1-\beta^2-\gamma^2 = a_3 + b_3 x + c_3 x^2, \tag{62 a}$$

where

$$a_3 = 1-\beta_2^2-\gamma_2^2 = \frac{m^2}{\varepsilon_-^2}, \tag{62 b}$$

$$b_3 = -2[\beta_2(\beta_1-\beta_2)+\gamma_2(\gamma_1-\gamma_2)] = \frac{2[(\varepsilon_+\varepsilon_- + m^2) y - m^2\omega^2]}{\varepsilon_-^2(\omega^2-y)}, \tag{62 c}$$

$$c_3 = -[(\beta_1-\beta_2)^2+(\gamma_1-\gamma_2)^2] = \frac{Y_3}{\varepsilon_-^2(\omega^2-y)^2}, \tag{62 d}$$

$$y = \varepsilon_+\varepsilon_- + m^2 - p_+ p_- \cos u, \tag{62 e}$$

$$Y_3 = (\omega^2 - p_+^2) y^2 - 2\omega^2(\varepsilon_+\varepsilon_- + m^2) y + m^2\omega^4. \tag{62 f}$$

The integration over the auxiliary variable x is now carried out, with the result

$$\int_0^{\pi} \sin \vartheta \, d\vartheta \int_0^{2\pi} d u_- \cdot F_1$$

$$= \frac{4\pi}{\alpha_1^2 \alpha_2^2} \Bigg\{ -\frac{1}{c_3} + \frac{1}{c_3} \log \frac{y}{\omega^2 - y} + \frac{b_3}{c_3^2} \log \frac{y}{\omega^2 - y} + \frac{1}{c_3} \log \frac{\varepsilon_-}{m} + \frac{b_3}{c_3^2} \log \frac{\varepsilon_-}{m} -$$

$$- \left(\frac{b_3}{2 c_3} + \frac{b_3^2}{2 c_3^2} - \frac{a_3}{c_3} \right) \frac{1}{\sqrt{-q_3}} l_+ + \frac{4}{q_3} - \frac{2(2 a_3 + b_3)}{q_3} \frac{1}{\sqrt{-q_3}} l_+ + 2 \beta_1^2 \Bigg[-\frac{2}{c_3^2} + \frac{2 a_3}{c_3 q_3} +$$

$$+ \left(\frac{1}{c_3^2} + \frac{2 b_3}{c_3^3} \right) \log \frac{y}{\omega^2 - y} + \left(\frac{1}{c_3^2} + \frac{2 b_3}{c_3^3} \right) \log \frac{\varepsilon_-}{m} -$$

$$- \left(\frac{b_3}{2 c_3^2} - \frac{q_3}{c_3^3} + \frac{2 a_3}{c_3^2} + \frac{a_3 (2 a_3 + b_3)}{c_3 q_3} \right) \frac{1}{\sqrt{-q_3}} l_+ + \frac{16 a_3}{q_3} + \frac{2(2 a_3 + b_3)}{q_3^2 (a_3 + b_3 + c_3)} -$$

$$- \frac{12 a_3 (2 a_3 + b_3)}{q_3^2} \frac{1}{\sqrt{-q_3}} l_+ \Bigg] \Bigg\} ;$$

$$q_3 = 4 a_3 c_3 - b_3^2. \tag{63}$$

The expression given in (63) should now be multiplied with $|\sin u|$ and integrated over all values of u from 0 to 2π. Instead of u we use the variable $y = \varepsilon_+ \varepsilon_- + m^2 - p_+ p_- \cos u$ as integration variable. For all the terms of (63) except those involving $\log [y/(\omega^2 - y)]$ the integration over y is elementary. We write

$$G_1 = G_1' + G_1'',$$

where G_1' is the part of G_1, involving only elementary integrals, and G_1'' is the part involving $\log [y/(\omega^2 - y)]$. After a straightforward but lengthy calculation we get the following form for G_1':

$$G_1' = \frac{4\pi (2 \varepsilon_+^2 + m^2)}{3 m^4 \omega^4 p_+^2} + \frac{12\pi}{\omega^4 p_+^2 p_-^2} - \frac{6\pi \varepsilon_+}{\omega^4 p_+^2 p_-^3} l_- +$$

$$+ \frac{4\pi (\varepsilon_-^2 + m^2)}{m^2 \omega^4 p_+^2 p_-^2} \log \frac{\varepsilon_-}{m} - \frac{4\pi \varepsilon_-}{\omega^4 p_+^2 p_-^3} l_- \cdot \log \frac{\varepsilon_-}{m} \cdot \tag{65}$$

As an example of the treatment of the terms in G_1'' we compute the term

$$J = 4\pi \int_0^{2\pi} |\sin u| \, d u \frac{1}{\alpha_1^2 \alpha_2^2} \frac{1}{c_3} \log \frac{y}{\omega^2 - y} = \frac{2\pi}{\omega^2 p_+ p_-} \int_{y_1}^{y_2} \frac{d y}{Y_3} \log \frac{y}{\omega^2 - y}, \tag{66}$$

where $y_1 = \varepsilon_+ \varepsilon_- + m^2 - p_+ p_-$, $y_2 = \varepsilon_+ \varepsilon_- + m^2 + p_+ p_-$. As new integration variable we introduce

$$t = \frac{y}{\omega^2 - y} \tag{67}$$

and get
$$J=\frac{2\pi}{\omega^4 p_+ p_-}\int_{t_1}^{t_2}\frac{dt}{T_3}\log t, \tag{68}$$
where
$$T_3=\varepsilon_-^2 t^2-2\,\varepsilon_+\varepsilon_- t+m^2=\varepsilon_-^2\,(t-t_3)\,(t-t_4);\ t_3=\frac{\varepsilon_+ + p_+}{\varepsilon_-};\ t_4=\frac{\varepsilon_+ - p_+}{\varepsilon_-}. \tag{69}$$
J can thus be written
$$J=\frac{\pi}{\omega^4\varepsilon_- p_+^2 p_-}\int_{t_1}^{t_2}\frac{dt}{t-t_3}\log t-\frac{\pi}{\omega^4\varepsilon_- p_+^2 p_-}\int_{t_1}^{t_2}\frac{dt}{t-t_4}\log t. \tag{70}$$

The integrals appearing in (70) are of the form $\int[\log t/(t+a)]$. All integrals of this form can be expressed in terms of a function of a single parameter, a possible standard form of which is the function
$$f(x)=\int_0^x\frac{-\log|1-t|}{t}\,dt \tag{71}$$
which has been studied and tabulated by Mitchell [12]. If J is expressed in terms of this function and elementary functions, the result is
$$\begin{aligned}J=\frac{\pi}{\omega^4\varepsilon_- p_+^2 p_-}&\left[\log t_3\log\frac{1+t_2/t_3}{1-t_1/t_3}-f(1-t_2/t_3)+f(1-t_1/t_3)\right]-\\&-\frac{\pi}{\omega^4\varepsilon_- p_+^2 p_-}\left[\log t_4\log\frac{1-t_2/t_4}{1-t_1/t_4}-f(1-t_2/t_4)+f(1-t_1/t_4)\right].\end{aligned} \tag{72}$$
The other terms of G_1'' are treated in a similar way, and the result is
$$\begin{aligned}G_1''=&\frac{2\pi\varepsilon_-}{\omega^4 p_+^2 p_-^3}\left[\log t_3\log\frac{1-t_2/t_3}{1-t_1/t_3}-f(1-t_2/t_3)+f(1-t_1/t_3)\right]-\\&-\frac{2\pi\varepsilon_-}{\omega^4 p_+^3 p_-^3}\left[\log t_4\log\frac{1-t_2/t_4}{1-t_1/t_4}-f(1-t_2/t_4)+f(1-t_1/t_4)\right]-\\&-\left(\frac{\pi t_3}{\omega^4 p_+ p_-^3}+\frac{\pi\varepsilon_-}{\omega^4 p_+^2 p_-^3}\right)\left[\frac{\log t_2}{1-t_2/t_3}-\frac{\log t_1}{1-t_1/t_3}-\log t_2+\log t_1+\log\frac{1-t_2/t_3}{1-t_1/t_3}\right]-\\&-\left(\frac{\pi t_4}{\omega^4 p_+ p_-^3}-\frac{3\pi\varepsilon_-}{\omega^4 p_+^2 p_-^3}\right)\left[\frac{\log t_2}{1-t_2/t_4}-\frac{\log t_1}{1-t_1/t_4}-\log t_2+\log t_1+\log\frac{1-t_2/t_4}{1-t_1/t_4}\right]-\\&-\frac{\pi t_3}{\omega^4 p_+ p_-^3}\left[-\frac{\log t_2}{(1-t_2/t_3)^2}+\frac{\log t_1}{(1-t_1/t_3)^2}+\log t_2-\log t_1+\frac{1}{1-t_1/t_3}-\frac{1}{1-t_1/t_4}-\right.\\&\left.-\log\frac{1-t_2/t_3}{1-t_1/t_3}\right]-\frac{\pi t_4}{\omega^4 p_+ p_-^3}\left[-\frac{\log t_2}{(1-t_2/t_4)^2}+\frac{\log t_1}{(1-t_1/t_4)^2}+\log t_2-\log t_1+\right.\\&\left.+\frac{1}{1-t_2/t_4}-\frac{1}{1-t_1/t_4}-\log\frac{1-t_2/t_4}{1-t_1/t_4}\right].\end{aligned} \tag{73}$$

We next turn to the computation of G_3. The same integration variables as in the integration of G_1 are used. We also introduce the notations (57) and

$$-p_+ \cdot k = \alpha_3 (1 + \beta_3' \cos u_- + \gamma_3' \sin u_-), \tag{74 a}$$

where

$$\beta_3' = \beta_3 \sin \vartheta; \quad \gamma_3' = \gamma_3 \sin \vartheta; \tag{74 b}$$

$$\alpha_3 = \varepsilon_+ \omega; \quad \beta_3 = \beta_+ \sin u; \quad \gamma_3 = -\beta_+ \cos u. \tag{74 c}$$

The integral over u_- is now of the form

$$\int_0^{2\pi} \frac{du_- \cos u_- (\cos u \cos u_- + \sin u \sin u_-)}{(1 + \beta_1' \cos u_- + \gamma_1' \sin u_-)^2 (1 + \beta_2' \cos u_- + \gamma_2' \sin u_-) (1 + \beta_3' \cos u_- + \gamma_3' \sin u_-)}$$

$$= -\frac{\partial}{\partial \beta_1'} \int_0^{2\pi} \frac{du_- (\cos u \cos u_- + \sin u \sin u_-)}{(1 + \beta_1' \cos u_- + \gamma_1' \sin u_-)(1 + \beta_2' \cos u_- + \gamma_2' \sin u_-)(1 + \beta_3' \cos u_- + \gamma_3' \sin u_-)}. \tag{75}$$

The integrand in (75) can be split up into three terms

$$\frac{\cos u \cos u_- + \sin u \sin u_-}{(1 + \beta_1' \cos u_- + \gamma_1' \sin u_-)(1 + \beta_2' \cos u_- + \gamma_2' \sin u_-)(1 + \beta_3' \cos u_- + \gamma_3' \sin u_-)}$$

$$= \frac{a_{23}'}{(1 + \beta_2' \cos u_- + \gamma_2' \sin u_-)(1 + \beta_3' \cos u_- + \gamma_3' \sin u_-)} +$$

$$+ \frac{a_{31}'}{(1 + \beta_3' \cos u_- + \gamma_3' \sin u_-)(1 + \beta_1' \cos u_- + \gamma_1' \sin u_-)} +$$

$$+ \frac{a_{12}'}{(1 + \beta_1' \cos u_- + \gamma_1' \sin u_-)(1 + \beta_2' \cos u_- + \gamma_2' \sin u)} \tag{76 a}$$

with

$$a_{23}' = \frac{a_{23}}{\sin \vartheta}, \quad a_{23} = \frac{-(\gamma_2 - \gamma_3) \cos u + (\beta_2 - \beta_3) \sin u}{\beta_3 (\gamma_2 - \gamma_1) + \beta_1 (\gamma_3 - \gamma_2) + \beta_1 (\gamma_1 - \gamma_3)}; \tag{76 b}$$

$$a_{31}' = \frac{a_{31}}{\sin \vartheta}, \quad a_{31} = \frac{-(\gamma_3 - \gamma_1) \cos u + (\beta_3 - \beta_1) \sin u}{\beta_3 (\gamma_2 - \gamma_1) + \beta_1 (\gamma_3 - \gamma_2) + \beta_2 (\gamma_1 - \gamma_3)}; \tag{76 c}$$

$$a_{12}' = \frac{a_{12}}{\sin \vartheta}, \quad a_{12} = \frac{-(\gamma_1 - \gamma_2) \cos u + (\beta_1 - \beta_2) \sin u}{\beta_3 (\gamma_2 - \gamma_1) + \beta_1 (\gamma_3 - \gamma_2) + \beta_2 (\gamma_1 - \gamma_3)}. \tag{76 d}$$

Making use of the identity (51), we can now integrate over u_- in the same way as in the treatment of G_1. After this integration we have

$$\int_0^{\pi} \sin \vartheta \, d\vartheta \int_0^{2\pi} du_- \cdot F_1 = -\frac{2\pi}{\alpha_1^2 \alpha_2 \alpha_3} \frac{\partial}{\partial \beta_1} \int_0^1 dx \int_0^{\pi} \sin \vartheta \, d\vartheta \cdot \frac{1}{\sin^2 \vartheta} \times$$

$$\times \left\{ \frac{a_{23}}{[1 - (\beta_{23}^2 + \gamma_{23}^2) \sin^2 \vartheta]^{3/2}} + \frac{a_{31}}{[1 - (\beta_{31}^2 + \gamma_{31}^2) \sin^2 \vartheta]^{3/2}} + \frac{a_{12}}{[1 - (\beta_{12}^2 + \gamma_{12}^2) \sin^2 \vartheta]^{3/2}} \right\}, \tag{77}$$

where $\beta_{23} = \beta_2 x + \beta_3 (1-x)$ etc. We cannot carry out the integration over ϑ in (77) directly, as each term gives a divergent integral. However, if we add -1 to every term, which we can do without altering the value of the sum since $a_{23} + a_{31} + a_{12} = 0$, the integrals become convergent

$$\int_0^\pi \frac{d\vartheta}{\sin\vartheta}\left[\frac{1}{[1-(\beta_{23}^2+\gamma_{23}^2)\sin^2\vartheta]^{3/2}} - 1\right] = \frac{2(\beta_{23}^2+\gamma_{23}^2)}{1-\beta_{23}^2-\gamma_{23}^2} - \log(1-\beta_{23}^2-\gamma_{23}^2). \qquad (78)$$

Adding $+2$ to each term, we now have

$$\begin{aligned}
&\int_0^\pi \sin\vartheta\, d\vartheta \int_0^\pi du_- \cdot F_3 \\
&\qquad = -\frac{2\pi}{\alpha_1^2\alpha_2\alpha_3}\frac{\partial}{\partial\beta_1}\int_0^1 dx \Biggl\{ a_{23}\left[\frac{2}{1-\beta_{23}^2-\gamma_{23}^2} - \log(1-\beta_{23}^2-\gamma_{23}^2)\right] + \\
&+ a_{31}\left[\frac{2}{1-\beta_{31}^2-\gamma_{31}^2} - \log(1-\beta_{31}^2-\gamma_{31}^2)\right] + a_{12}\left[\frac{2}{1-\beta_{12}^2-\gamma_{12}^2} - \log(1-\beta_{12}^2-\gamma_{12}^2)\right]\Biggr\}.
\end{aligned} \qquad (79)$$

The differentiation with respect to β_1 and the integration over the auxiliary variable x are straightforward, and we need not go through the details here. The result is

$$\begin{aligned}
&\int_0^\pi \sin\vartheta\, d\vartheta \int_0^{2\pi} du_- \cdot F_3 \\
&= -\frac{2\pi}{\alpha_1^2\alpha_2\alpha_3}\Biggl\{\frac{\gamma_2-\gamma_3}{D}\Biggl(a_{23}\Biggl[\left(2+\frac{b_1}{c_1}\right)\log\frac{\varepsilon_-}{m} - \frac{b_1}{c_1}\log\frac{\varepsilon_+}{m} + \left(2-\frac{q_1}{2c_1}\right)\frac{1}{\sqrt{-q_1}}\cdot\Lambda\Biggr] + \\
&\quad + a_{31}\Biggl[\left(2+\frac{b_2}{c_2}\right)\log\frac{\varepsilon_+}{m} + \frac{b_2}{c_2}\log\frac{y}{\omega^2-y} + \left(2-\frac{q_2}{2c_2}\right)\frac{1}{\sqrt{-q_2}}\,l_-\Biggr] + \\
&\quad + a_{12}\Biggl[-\left(2+\frac{b_3}{c_3}\right)\log\frac{y}{\omega^2-y} - \frac{b_3}{c_3}\log\frac{\varepsilon_-}{m} + \left(2-\frac{q_3}{2c_3}\right)\frac{1}{\sqrt{-q_3}}\,l_+\Biggr]\Biggr) + \\
&+ \frac{\sin u}{D}\Biggl(-\left(2+\frac{b_3}{c_3}\right)\log\frac{y}{\omega^2-y} - \frac{b_3}{c_3}\log\frac{\varepsilon_-}{m} + \left(2-\frac{q_3}{2c_3}\right)\frac{1}{\sqrt{-q_3}}\,l_+ - \\
&\quad - \left(2+\frac{b_2}{c_2}\right)\log\frac{\varepsilon_+}{m} - \frac{b_2}{c_2}\log\frac{y}{\omega^2-y} - \left(2-\frac{q_2}{2c_2}\right)\frac{1}{\sqrt{-q_2}}\,l_-\Biggr) + \\
&+ a_{31}\Biggl(4\beta_3\left[\frac{2}{q_2} - \frac{2a_2+b_2}{q_2}\frac{1}{\sqrt{-q_2}}\,l_-\right] + 4\beta_1\left[-\frac{2a_2+b_2}{a_2q_2} + \frac{2(a_2+b_2+c_2)}{q_3}\frac{1}{\sqrt{-q_2}}\,l_-\right] + \\
&\quad + 2\beta_3\Biggl[-\frac{1}{c_2} - \left(\frac{1}{c_2}+\frac{b_2}{c_2^2}\right)\log\frac{\varepsilon_+}{m} - \left(\frac{1}{c_2}+\frac{b_2}{c_2^2}\right)\log\frac{y}{\omega^2-y} - \frac{b_2^2+b_2c_2-2a_2c_2}{2c_2^2}\frac{1}{\sqrt{-q_2}}\,l_-\Biggr] +
\end{aligned}$$

$$+2\beta_1\left[\frac{1}{c_2}+\left(\frac{2}{c_2}+\frac{b_2}{c_2^2}\right)\log\frac{\varepsilon_+}{m}+\left(\frac{2}{c_2}+\frac{b_2}{c_2^2}\right)\log\frac{y}{\omega^2-y}+\right.$$

$$\left.+\left(1+\frac{b_2}{c_2}+\frac{b_2^2-2a_2c_2}{2c_2^2}\right)\frac{1}{\sqrt{-q_2}}l_-\right]\Bigg)$$

$$+a_{12}\Bigg(4\beta_1\left[-\frac{2a_3+b_3}{q_3(a_3+b_3+c_3)}+\frac{2a_3}{q_3}\frac{1}{\sqrt{-q_3}}l_+\right]+$$

$$+2\beta_1\left[\frac{1}{c_3}-\frac{b_3}{c_3^2}\log\frac{y}{\omega^2-y}-\frac{b_3}{c_3^2}\log\frac{\varepsilon_-}{m}+\frac{b_3^2-2a_3c_3}{2c_3^2}\frac{1}{\sqrt{-q_3}}l_+\right]\Bigg)\Bigg\}, \qquad (80)$$

where

$$D=\beta_3(\gamma_2-\gamma_1)+\beta_1(\gamma_3-\gamma_2)+\beta_2(\gamma_1-\gamma_3)=\frac{p_+p_-y\sin u}{\varepsilon_+\varepsilon_-(\omega^2-y)},$$

$$\Lambda=\log\frac{\varepsilon_+\varepsilon_--p_+p_-\cos u+\sqrt{(\varepsilon_+\varepsilon_--p_+p_-\cos u)^2-m^4}}{\varepsilon_+\varepsilon_--p_+p_-\cos u-\sqrt{(\varepsilon_+\varepsilon_--p_+p_-\cos u)^2-m^4}},$$

$$a_1=\frac{m^2}{\varepsilon_+^2},$$

$$b_1=\frac{2}{\varepsilon_+^2\varepsilon_-}[\varepsilon_+(\varepsilon_+\varepsilon_--p_+p_-\cos u)-\varepsilon_-m^2],$$

$$c_1=\frac{1}{\varepsilon_+^2\varepsilon_-^2}[(\varepsilon_+^2+\varepsilon_-^2)m^2-2\varepsilon_+\varepsilon_-(\varepsilon_+\varepsilon_--p_+p_-\cos u)],$$

$$q_1=-\frac{4}{\varepsilon_+^2\varepsilon_-^2}([\varepsilon_+\varepsilon_--p_+p_-\cos u)^2-m^4],$$

$$a_2=\frac{y^2}{(\omega^2-y)^2},$$

$$b_2=\frac{2\omega y(\varepsilon_-\omega-y)}{\varepsilon_+(\omega^2-y)^2},$$

$$c_2=\frac{Y_2}{\varepsilon_+^2(\omega^2-y)^2},$$

$$Y_2=(\omega^2-p_-^2)y^2-2\omega^2(\varepsilon_+\varepsilon_-+m^2)y+m^2\omega^4,$$

$$q_2=-\frac{4p_-^2y^2}{\varepsilon_+^2(\omega^2-y)^2},$$

and a_3, b_3, and c_3 are defined in (62). The remaining integration is performed in the same manner as for G_1 by introducing the variable y, also defined in (62). Most of the terms of (80) give rise to elementary integrals, the only exceptions being the terms involving $\log[y/(\omega^2-y)]$, and the terms involving Λ.

The first kind of terms is treated in the manner outlined for the corresponding terms of G_1 by introducing the variable $t = y/(\omega^2 - y)$. The integrals can be expressed in terms of elementary functions and f-functions. The term involving Λ is treated in the following way. We introduce the variable

$$x = \frac{\varepsilon_+ \varepsilon_- - p_+ p_- \cos u}{m^2}. \tag{81}$$

The term under consideration is then equal to

$$-2\pi \int_0^{2\pi} d\,u \,|\sin u|\, \frac{\gamma_2 - \gamma_3}{\alpha_1^2 \alpha_2 \alpha_3} \frac{a_{23}}{D} \left(2 - \frac{q_1}{2\,c_1}\right) \frac{1}{\sqrt{-q_1}}\,\Lambda$$

$$= \frac{\pi}{m^2 \omega^2 p_+^2 p_-^2} \int_{x_1}^{x_2} \frac{d\,x\,[\varepsilon_+ \varepsilon_- x^2 - (\varepsilon_+^2 + \varepsilon_-^2)\,x + \varepsilon_+ \varepsilon_-]}{(x+1)^2 (\varepsilon_+^2 + \varepsilon_-^2 - 2\,\varepsilon_+ \varepsilon_- x)\sqrt{x^2 - 1}} \log \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}}. \tag{82}$$

By means of a new transformation of the variable

$$z = x + \sqrt{x^2 - 1} \tag{83}$$

we bring the integral into a form that tells us that this term also can be expressed by means of elementary functions and f-functions. The result of the very tedious calculation of all the terms of G_3 is

$$\begin{aligned}
G_3 = &- \frac{4\pi\,(2\,\varepsilon_+ \varepsilon_- + m^2)}{3\,m^4 \omega^4 p_+ p_-} - \left(\frac{2\pi\, p_-}{3\,m^2 \omega^5 p_+^2} - \frac{2\pi\,(\varepsilon_+^2 + \varepsilon_-^2 - \varepsilon_+ \varepsilon_-)}{m^2 \omega^5 p_+^2 p_-}\right) l_+ - \\
&- \left(\frac{2\pi\, p_+}{3\,m^2 \omega^5 p_-^2} - \frac{2\pi\,(\varepsilon_+^2 + \varepsilon_-^2 - \varepsilon_+ \varepsilon_-)}{m^2 \omega^5 p_+ p_-^2}\right) l_- + \frac{2\pi\,\varepsilon_+ (\varepsilon_+ - \varepsilon_-)}{m^2 \omega^5 p_+^2 p_-}\, l_+ \log \frac{\varepsilon_+}{m} + \\
&+ \frac{2\pi\,\varepsilon_- (\varepsilon_- - \varepsilon_+)}{m^2 \omega^5 p_+ p_-^2}\, l_- \log \frac{\varepsilon_-}{m} - \frac{4\pi\,\varepsilon_+}{m^2 \omega^5 p_+ p_-} \log \frac{\varepsilon_+}{m} - \frac{4\pi\,\varepsilon_-}{m^2 \omega^5 p_+ p_-} \log \frac{\varepsilon_-}{m} - \\
&- \frac{2\pi\,\varepsilon_+ \varepsilon_- (\varepsilon_+ - \varepsilon_-)}{m^2 \omega^5 p_+^2 p_-^2}\, L \log \frac{\varepsilon_+}{m} - \frac{2\pi\,\varepsilon_+ \varepsilon_- (\varepsilon_- - \varepsilon_+)}{m^2 \omega^5 p_+^2 p_-^2}\, L \log \frac{\varepsilon_-}{m} - \\
&- \frac{4\pi\,\varepsilon_+ \varepsilon_- (\varepsilon_+ - \varepsilon_-)}{m^2 \omega^5 p_+^2 p_-^2} \log \left|\frac{\beta_+ + \beta_-}{\beta_+ - \beta_-}\right| \log \frac{\varepsilon_+}{m} - \\
&- \frac{4\pi\,\varepsilon_+ \varepsilon_- (\varepsilon_- - \varepsilon_+)}{m^2 \omega^5 p_+^2 p_-^2} \log \left|\frac{\beta_+ + \beta_-}{\beta_+ - \beta_-}\right| \log \frac{\varepsilon_-}{m} + \frac{\pi\,(\varepsilon_+ - \varepsilon_-)}{\omega^5 p_+^2 p_-^2 t_3}\, T_1(t_3) + \\
&+ \frac{\pi\,(\varepsilon_+ - \varepsilon_-)}{\omega^5 p_+^2 p_-^2 t_4}\, T_1(t_4) + \frac{\pi\,(\varepsilon_- - \varepsilon_+)}{\omega^5 p_+^2 p_-^2 t_3'}\, T_1(t_3') + \frac{\pi\,(\varepsilon_- - \varepsilon_+)}{\omega^5 p_+^2 p_-^2 t_4'}\, T_1(t_4') + \\
&+ \frac{\pi}{\omega^5 p_+^2 p_-}\, T_2(t_3) - \frac{\pi}{\omega^5 p_+^2 p_-}\, T_2(t_4) + \frac{\pi}{\omega^5 p_+ p_-^2}\, T_2(t_3') - \frac{\pi}{\omega^5 p_+ p_-^2}\, T_2(t_4') -
\end{aligned}$$

$$-\frac{4\pi(\varepsilon_+^2+\varepsilon_-^2)}{m^2\omega^4 p_+^2 p_-^2}Z_2+\frac{8\pi}{m^2\omega^2 p_+^2 p_-^2}Z_3-\frac{8\pi}{m^2\omega^2 p_+^2 p_-^2}Z_4+\frac{2\pi\varepsilon_+\varepsilon_-(\varepsilon_+-\varepsilon_-)}{m^2\omega^5 p_+^2 p_-^2}\times$$

$$\times\left[\log\frac{\varepsilon_+}{\varepsilon_-}\log\frac{1-(\varepsilon_-/\varepsilon_+)z_2}{1-(\varepsilon_-/\varepsilon_+)z_1}-f\left(1-\frac{\varepsilon_-}{\varepsilon_+}z_2\right)+f\left(1-\frac{\varepsilon_+}{\varepsilon_-}z_1\right)\right]+$$

$$+\frac{2\pi\varepsilon_+\varepsilon_-(\varepsilon_--\varepsilon_+}{m^2\omega^5 p_+^2 p_-^2}\times$$

$$\times\left[\log\frac{\varepsilon_-}{\varepsilon_+}\log\frac{1-(\varepsilon_+/\varepsilon_-)z_2}{1-(\varepsilon_+/\varepsilon_-)z_1}-f\left(1-\frac{\varepsilon_+}{\varepsilon_-}z_2\right)+f\left(1-\frac{\varepsilon_+}{\varepsilon_-}z_1\right)\right], \tag{84}$$

where we have introduced the following, notations besides those used in (65) and (73)

$$t_3'=\frac{\varepsilon_-+p_-}{\varepsilon_+};\quad t_4'=\frac{\varepsilon_--p_-}{\varepsilon_+}; \tag{85 a}$$

$$T_i(t_3)=\int_{t_1}^{t_2}\frac{dt}{(t-t_3)^i}\log t, \tag{85 b}$$

$$Z_i=\int_{z_1}^{z_2}\frac{dz}{(z+1)^i}\log z. \tag{85 c}$$

The integrals (85 b), (85 c) give non-elementary functions only for $i=1$, e.g.

$$T_1(t_3)=\log t_3\log\frac{1-t_2/t_3}{1-t_1/t_3}-f(1-t_2/t_3)+f(1-t_1/t_3). \tag{86}$$

The remaining terms of I_3'', G_4, G_5, G_6 are computed in an analogous way, and thus we have no reasons to give the details of the calculation, the result of which is

$$G_4=\frac{8\pi}{\omega^2 p_+^2 p_-^2}-\frac{4\pi(\varepsilon_+\varepsilon_-+m^2)}{\omega p_+^3 p_-^2(\omega^2-p_+^2)}l_++\frac{4\pi\varepsilon_-}{\omega^2 p_-^3(\omega^2-p_+^2)}l_-+$$

$$+\frac{8\pi\varepsilon_-}{\omega p_+^2 p_-^2(\omega^2-p_+^2)}\log\frac{\varepsilon_-}{m}+\left(\frac{8\pi\varepsilon_-(\varepsilon_+\varepsilon_-+m^2)}{\omega^2 p_-^3(\omega^2-p_+^2)^2}-\frac{4\pi\varepsilon_-^2}{\omega p_+^2 p_-^3(\omega^2-p_+^2)}\right)l\cdot\log\frac{\varepsilon_-}{m}+$$

$$+\left(\frac{4\pi}{\omega^2 p_+p_-(\omega^2-p_+^2)}-\frac{8\pi\varepsilon_-^2}{p_+p_-^3(\omega^2-p_+^2)^2}\right)L\cdot\log\frac{\varepsilon_-}{m}-\frac{4\pi(\varepsilon_+\varepsilon_-+m^2)}{p_+^3p_-^3(\omega^2-p_+^2)}\cdot L+$$

$$+\left(\frac{4\pi}{\omega^2 p_+p_-(\omega^2-p_+^2)}-\frac{4\pi\varepsilon_-^2}{p_+p_-^3(\omega^2-p_+^2)^2}\right)l_+\cdot l_-+$$

$$+\left(\frac{2\pi p_-}{\omega^2\varepsilon_-p_+^2(\omega^2-p_+^2)}+\frac{2\pi m^2\omega}{p_+^4p_-^3(\omega^2-p_+^2)}+\frac{4\pi\varepsilon_-(\varepsilon_+\varepsilon_-+m^2)}{p_+^2p_-^3(\omega^2-p_+^2)^2}\right)L\cdot l_-+$$

$$+\frac{4\pi[(\varepsilon_+\varepsilon_-+m^2)^2+\varepsilon_-^2\omega^2]}{\omega^2 p_+ p_-^3(\omega^2-p_+^2)^2}\times\left[f\left(\frac{\omega^2}{\omega^2-y_1}\right)-f\left(\frac{\omega^2}{\omega^2-y_2}\right)\right]+$$

$$+\left(\frac{2\pi\varepsilon_-^2}{\omega^2 p_+^2 p_-^3(\omega+p_+)}+\frac{2\pi\varepsilon_- m^2 t_3}{\omega^2 p_+^2 p_-^3(\omega+p_+)^2}-\frac{2\pi\varepsilon_-^2\varepsilon_+ t_3^2}{\omega^2 p_+^2 p_-^3(\omega+p_+)^2}\right)T_1(t_3)-$$

$$-\left(\frac{2\pi\varepsilon_-^2}{\omega^2 p_+^2 p_-^3(\omega-p_+)}+\frac{2\pi\varepsilon_- m^2 t_4}{\omega^2 p_+^2 p_-^3(\omega-p_+)^2}-\frac{2\pi\varepsilon_-^2\varepsilon_+ t_4^2}{\omega^2 p_+^2 p_-^3(\omega-p_+)^2}\right)T_1(t_4)+$$

$$+\frac{2\pi\varepsilon_- t_3^2}{\omega^2 p_+ p_-^3(\omega+p_+)}T_2(t_3)+\frac{2\pi\varepsilon_- t_4^2}{\omega^2 p_+ p_-^3(\omega-p_+)}T_2(t_4) \tag{87}$$

and

$$G_{[illegible]}=\frac{4\pi\varepsilon_-}{\omega^3 p_-^2(\omega^2-p_+^2)}L\cdot\log\frac{\varepsilon_-}{m}+\frac{4\pi\varepsilon_+}{\omega^3 p_+^2(\omega^2-p_-^2)}L\cdot\log\frac{\varepsilon_+}{m}-$$

$$-\frac{2\pi(\varepsilon_+\varepsilon_-+m^2)}{\omega(\omega^2-p_+^2)p_+^3 p_-^2}L\cdot l_+-\frac{2\pi(\varepsilon_+\varepsilon_-+m^2)}{\omega p_+^2 p_-^3(\omega^2-p_-^2)}L\cdot l_-+$$

$$+\frac{4\pi\varepsilon_-}{\omega^2(\omega^2-p_+^2)p_+ p_-^2}l_-\cdot\log\frac{\varepsilon_-}{m}+\frac{4\pi\varepsilon_+}{\omega^2(\omega^2-p_-^2)p_+^2 p_-}l_+\cdot\log\frac{\varepsilon_+}{m}+$$

$$+\frac{2\pi\varepsilon_-}{\omega^3 p_-^2(\omega^2-p_+^2)}l_+\cdot l_-+\frac{2\pi\varepsilon_+}{\omega^3 p_+^2(\omega^2-p_-^2)}l_+\cdot l_+-$$

$$-\frac{4\pi(\varepsilon_+-\varepsilon_-)}{\omega^3 p_+^2 p_-^2}\log\left|\frac{\beta_++\beta_-}{\beta_+-\beta_-}\right|\log\frac{\varepsilon_-}{m}-\frac{4\pi(\varepsilon_--\varepsilon_+)}{\omega^3 p_+^2 p_-^2}\log\left|\frac{\beta_++\beta_-}{\beta_+-\beta_-}\right|\log\frac{\varepsilon_+}{m}+$$

$$+\left(\frac{4\pi}{\omega^2 p_+^2 p_-^2}-\frac{4\pi\varepsilon_-}{\omega(\omega^2-p_-^2)p_+^2 p_-^2}-\frac{4\pi\varepsilon_+}{\omega(\omega^2-p_+^2)p_+^2 p_-^2}\right)\times$$

$$\times\left[f\left(\frac{\omega^2}{\omega^2-y_1}\right)-f\left(\frac{\omega^2}{\omega^2-y_2}\right)\right]-\frac{2\pi\varepsilon_-}{\omega^3(\omega+p_+)p_+ p_-^2}T_1(t_3)+$$

$$+\frac{2\pi\varepsilon_-}{\omega^3(\omega-p_+)p_+ p_-^2}T_1(t_4)-\frac{2\pi\varepsilon_+}{\omega^3(\omega+p_-)p_+^2 p_-}T_1(t_3')+\frac{2\pi\varepsilon_+}{\omega^3(\omega-p_-)p_+^2 p_+}T_1(t_4')+$$

$$+\frac{2\pi}{\omega^2 p_+^2 p_-^2}[\log^2 z_2-\log^2 z_1]-\frac{8\pi}{\omega^2 p_+^2 p_-^2}Z_2-$$

$$-\frac{2\pi(\varepsilon_+-\varepsilon_-)}{\omega^3 p_+^2 p_-^2}\left[\log\frac{\varepsilon_+}{\varepsilon_-}\log\frac{1-\varepsilon_- z_2/\varepsilon_+}{1-\varepsilon_- z_1/\varepsilon_+}-f\left(1-\frac{\varepsilon_-}{\varepsilon_+}z_2\right)+f\left(1-\frac{\varepsilon_-}{\varepsilon_+}z_1\right)\right]-$$

$$-\frac{2\pi(\varepsilon_--\varepsilon_+)}{\omega^3 p_+^2 p_-^2}\left[\log\frac{\varepsilon_-}{\varepsilon_+}\log\frac{1-\varepsilon_+ z_2/\varepsilon_-}{1-\varepsilon_+ z_1/\varepsilon_-}-f\left(1-\frac{\varepsilon_+}{\varepsilon_-}z_2\right)+f\left(1-\frac{\varepsilon_+}{\varepsilon_-}z_1\right)\right]. \tag{88}$$

From the formulae (65), (73), (84), (87), and (88) the value of I_3'' is given.

VI. Numerical results

We have now completed the integration over the angular variables. What we are interested in is the value of β we arrive at if we integrate also over all the values of the energy of one of the outgoing electrons. The result cannot be expressed in terms of tabulated functions. We have therefore done the integration over $d\varepsilon_+$ numerically. The value used for the mass of the π^0 is 264 m_e [13], and thus the energy ω of each photon is 132 m_e. We have multiplied I_1 and I_3 with $p_+ p_-$, and normalized them so that the curve representing I_1 gives the Bethe-Heitler cross section in units $\bar{\phi} = \alpha Z^2 r_0^2$. The curves are given as functions of the energy of the outgoing positron, or, more precisely, as functions of the kinetic energy of the positron, divided by the total available energy, i. e. $\eta = (\varepsilon_+ - m)/(\omega - 2m)$. As the curves are symmetric in + and − we give the values only for $0 \leqslant \eta \leqslant 0.5$, or $m_e \leqslant \varepsilon_+ \leqslant 66\, m_e$ in Fig. 3 and Table 1.

The integration over $d\varepsilon_+$ is done in the following way. We use Simpson's rule to determine the area under the curves from $\eta = \eta_1$ to $\eta = 0.5$. For each

Table 1. Bethe-Heitler cross section ($\sigma_{\text{B.-H.}}$) and $-\beta\,\sigma_{\text{B.-H.}}$.

$\eta = \dfrac{\varepsilon_+ - m}{\omega - 2m}$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$\sigma_{\text{B.-H.}}/\bar{\phi}$	7.80	9.33	9.88	10.04	10.04	9.96	9.86	9.77	9.71	9.69
$-\beta\,\sigma_{\text{B.-H.}}/\bar{\phi}$	0.00	0.00	0.02	0.23	0.57	1.03	1.49	1.92	2.29	2.41

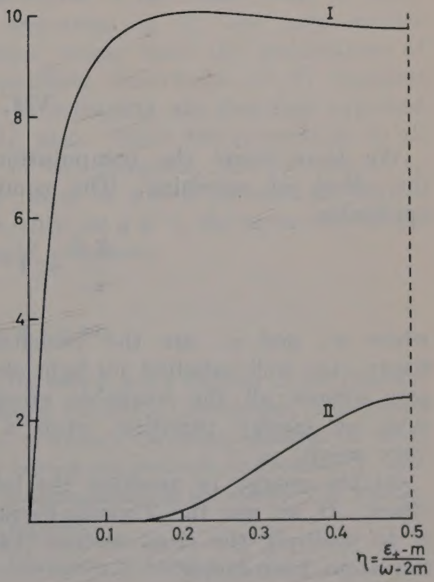

Fig. 3. Bethe-Heitler cross section $\sigma_{\text{B.-H.}}$ for $\omega = 132\, m_e$ in units $\bar{\phi}$ (I), and $-\beta\,\sigma_{\text{B.-H.}}$ in same units (II).

value of η_1 the quotient between the area from (II) and the area from (I) is taken. The square of this quotient is given in Table 2 and in Fig. 4 as a function of η_1. $\beta^2(\eta_1)$ is thus the value we obtain for β^2 if we take into account only those cases, where both the outgoing electrons have a kinetic energy larger than $\eta_1(\omega - 2m) = 130\ \eta_1$.

Table 2. The coefficient β^2.

η_1	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
β^2	0.0091	0.0102	0.0125	0.0163	0.0218	0.0289	0.0373	0.0449	0.0530	0.0595	0.0619

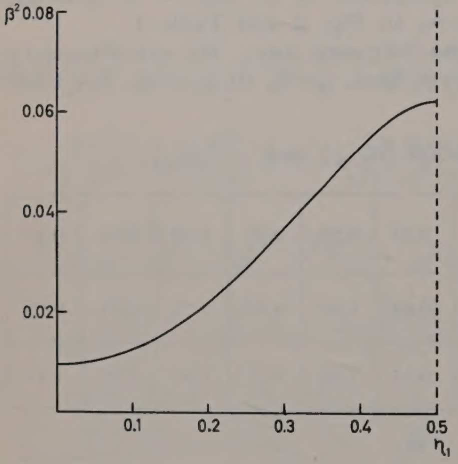

Fig. 4. The integrated angular correlation coefficient β^2 as a function of the smallest energy observed.

VII. Corrections

We have made the computation in the Born approximation, and neglected the effect of screening. The conditions for the Born approximation to be applicable

$$\frac{Ze^2}{v_+} \ll 1 \quad \text{and} \quad \frac{Ze^2}{v_-} \ll 1, \tag{89}$$

where v_+ and v_- are the velocities of the positon and the negaton, respectively, are well satisfied for light elements, except when one of outgoing particles gets almost all the available energy. As both I_1 and I_3 are very small for such an energy partition, even a large correction here will not change β^2 very much.

At the energy in question the influence of screening is a much more serious effect. If we use the Thomas-Fermi model of the atom, the effect of screening is to multiply the cross section (14) with the factor $(1 - F(q))^2$, where $F(q)$ is the atom form-factor

$$F(q) = \frac{1}{Z} \int \varrho(r)\, e^{i\bar{q}\cdot\bar{r}}\, dV, \tag{90}$$

$\varrho(r)$ being the density of atomic electrons at a distance r from the nucleus according to Fermi's model. $F(q)$ becomes comparable to 1 when q is of the same order of magnitude as the inverse atomic radius. We get the smallest value of q when all momenta are parallel to each other

$$q_{\min} = \omega - p_+ - p_- = \frac{m^2}{\varepsilon_+ + p_+} + \frac{m^2}{\varepsilon_- + p_-} \simeq \frac{\omega m^2}{2\,\varepsilon_+\,\varepsilon_-}, \tag{91}$$

the last equality holding approximately when all energies are large compared to m_e. In our case we get $q_{\min} = m/66 = 137/66\, a_0$ when $\varepsilon_+ = \varepsilon_-$. For this value of q, $F(q)$ is about 0.26 for hydrogen (see e.g. [14] for a table of $F(q)$), and the neglect of screening seems to be a very serious drawback. However, the effect of screening is certainly very much smaller than the value of $q_{\min}$ might indicate. The influence of screening on the total cross section for pair production has been calculated by Bethe [15]. He finds that the main part of the pair production is connected with a momentum transfer q between $q_{\min}$ and m_e, the probability $\phi(q)$ being proportional to $1/q$ in this region. The result of this is that the correction due to screening becomes fairly small for the energies we are interested in. The largest correction to the cross section comes when $\varepsilon_+ = \varepsilon_-$ and amounts to about 4 % for hydrogen. The total cross section is about 3 % less than it would be if screening were not taken into account (if liquid hydrogen is used in the bubble chamber). Now the integrands appearing in I_3' are exactly the same as some of the integrals appearing in I_1, and it can easily be shown that the q-dependence here also is of the form $1/q$ in the region of q where the contributions to the integral are appreciable. This means that the correction due to screening is of the same order of magnitude for I_3' as for I_1. The integrands appearing in I_3'' are very similar to those appearing in I_3', the only difference being that the numerators of I_3'' contain e.g. $\cos^2 u_+$, while the corresponding numerator of I_3' contains $\cos^2 u_+ + \sin^2 u_+ \cos^2 \vartheta$. The corrections due to screening are therefore expected to be of the same order of magnitude for I_3'' also. Since the corrections to all the integrals diminish the absolute values of the integrals, the corrections to the computed value of β^2 is at most of the order of a few per cent for hydrogen. As the atom form factor $F(q)$ depends only on $q\,Z^{-\frac{1}{3}}$, the error due to the approximations used is not very large for light elements.

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